Determination of the Preliminary Orbits of Artificial Lunar and Planetary Satellites

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Preliminary orbit deteimination theories for artificial lunar and planetaiy satellites aie developed in this paper The theories are based on Doppler obseivations alone Both circular and elliptical preliminary orbits are treated The portions of classical spectioscopic binaiy star theory that are applicable to the pioblem are pointed out The secular bulge peituibations, which accumulate during the observation interval, are taken into account in the development These terms heretofore have been neglected in pieliminary orbit determination theories for artificial planetary satellites It is important that they be included if the theory is to have practical application Particular attention is directed to lunar and Martian artificial satellite orbits, but the theories aie applicable to satellites of other planets as well A numerical verification of the theory for a lunar satellite reveals that the observations used in the orbit determination must be selected with considerable care Also, moie than the minimum number of observations and a least-squares solution foi the elements are necessaiy in order to obtain satisfactory results

Nomenclature

a = the semimajor axis of an orbit; when used with a subscript, it denotes the equatorial radius of a planet
A = an auxiliary quantity, defined by Eq (22)
$\mathbf{A}$ = an auxiliary vector, defined by Eq (79)
$\mathbf{B}$ = an auxiliary vector, defined by Eq (80)
$\mathbf{C}$ = an auxiliary vector, defined by Eq (23) or Eq (31)
$\mathbf{D}$ = an auxiliary vector, defined by Eq (24) or Eq (32)
e = the eccentricity of an orbit
E = the eccentric anomaly
f = a coefficient in the "f and g expressions
g = a coefficient in the "f and g ' expressions
i = the inclination of the orbit plane to a fundamental plane
J_2 = the coefficient of the second zonal harmonic in a planet's gravitational potential
k = the planetocentric gravitational constant
m = mass
M = the mean anomaly
n = the satellite's mean motion
p = the semilatus rectum of an orbit
P = the satellite's orbital period of revolution
$\mathbf{P}$ = the unit vector directed to perifocus
$\mathbf{Q}$ = the unit vector completing the right handed orthogonal set with $\mathbf{P}$ and $\mathbf{W}$
r = $|\mathbf{r}|$ = the satellite's radial distance from the primary focus
$\dot{r}$ = the satellite s radial velocity; $\dot{r} \neq |\dot{\mathbf{r}}|$
$\mathbf{r}$ = the radius vector from the primary focus to the satellite
$\dot{\mathbf{r}}$ = the satellite's velocity
$\mathbf{r}_T$ = the geocentric position of the observer
$\dot{\mathbf{r}}_T$ = the geocentric velocity of the observer
$\mathbf{S}$ = an auxiliary vector, defined by Eq (64)
t = time as measured in conventional units
T = the time of perifocal passage
T_Ω = the time of crossing the ascending node
$\mathbf{T}$ = an auxiliary vector, defined by Eq (65)
u = the true argument of latitude ($u = v + \omega$)
$\mathbf{U}$ = the radial unit vector, along $\mathbf{r}$
v = the true anomaly
$\mathbf{V}$ = the unit vector completing the right-handed orthogonal set with $\mathbf{U}$ and $\mathbf{W}$
$\mathbf{W}$ = the unit vector normal to the orbit plane
x_ω = the component of $\mathbf{r}$ along $\mathbf{P}$
y_ω = the component of $\mathbf{r}$ along $\mathbf{Q}$
μ = a mass function, numerically equal to the sum of the masses of the satellite and primary body in terms of the mass of the primary body as unity
ρ = $|\boldsymbol{\varrho}|$ = the distance of the satellite from the observer
$\dot{\rho}$ = the radial velocity of the satellite with respect to the observer, that is, the range rate; $\dot{\rho} \neq |\dot{\boldsymbol{\varrho}}|$
$\dot{\rho}^*$ = the range rate, as corrected for the observer's diurnal motion, etc
$\boldsymbol{\varrho}$ = the position of the satellite with respect to the obseiver
τ = the characteristic unit of time; $\tau = k(t - t_0)$
γ = the vernal equinox
ω = the argument of perifocus
Ω = the longitude of the ascending node

Subscripts

0 = epoch
p = planet
$\oplus$ = Earth
♂ = Mars
☾ = moon

Superscript

(ˋ) = perturbative vaiiation with respect to τ

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I Introduction

PRELIMINARY orbit determination methods foi artificial lunar and planetary satellites are investigated in this papei Aitificial planetary satellites will not be visible by reflected sunlight For example, a specularly reflecting spherical Martian satellite of 10-ft radius and 100% reflectivity will be at 26 ^{m}8 when Mars is at opposition This is, of course, far beyond the limiting magnitude of any existing telescope Even a lunar balloon satellite would have to be extiemely large in order for optical observation to be possible [13] In view of the faintness of these objects then, and the lack of accurate angle and range observations, the preliminary orbit determination methods considered herein are based on Doppler data alone

The pioblem is thus somewhat similar to that of determining the oibits of spectroscopic binary stars In fact, the classical methods used for spectroscopic binaiies, assuming unperturbed or two-body motion, have already been adapted to certain portions of the problem [6 7 9]

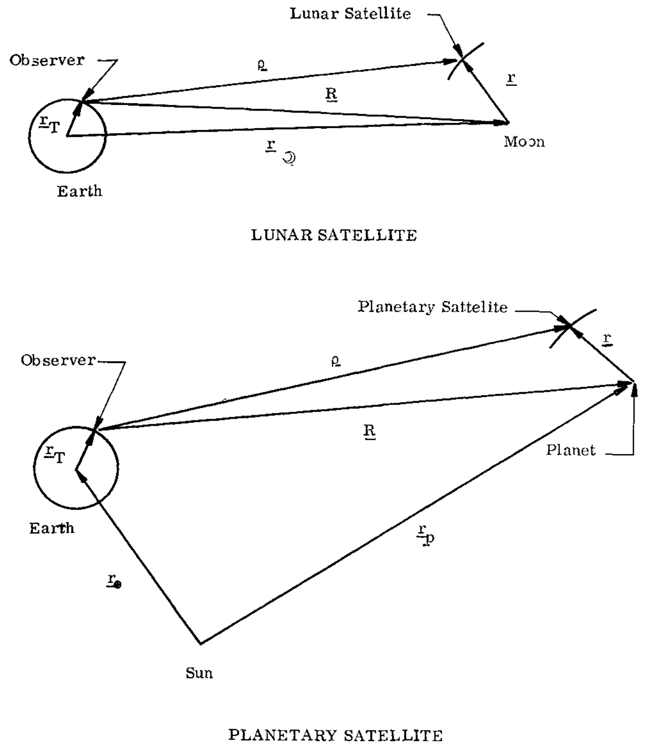

Fig 1 Schematic of vector relationships for lunar and planetary satellites

The Doppler or range-rate observations used in the preliminary orbit determination scheme may have to extend over a number of days foı a lunar satellite, and perhaps several months for a planetary satellite The long observation interval is necessary in order that the orbit can be viewed from substantially different directions If the orbit is seen from only one direction, the problem reduces to exactly that of the spectroscopic binaries In this case, the longitude of the ascending node and the quadrant of the inclination cannot be determined For a lunar satellite, crude angle observations that tell no more than on which side of the moon the satellite is located may prove helpful in resolving the node and inclination difficulty without waiting several days Also, Hanaway and Myeıs[9] indicate that observations of occultation times by several Earth stations may be used for this purpose

The satellite's period of revolution can be found very accurately from its radial velocity curve, provided the satellite encounters no atmospheric drag The shape and amplitude of the radial velocity curve will vary with time since the direction of the satellite system will be continually changing, but this will not affect the accurate determination of the period

Having found the period P from the Doppler curve, the mean motion n is immediately available from

$$n = 2\pi/P \tag{1}$$

If the mass of the planetary body is known, Kepler's harmonic law gives the semimajor axis of the orbit:

$$a = [k\mu^{1/2}P/2\pi]^{2/3} \tag{2}$$

where k is the planetocentric gravitational constant and μ is numerically the sum of the masses of the satellite and primary body in terms of the mass of the primary body which is taken as unity For example, for an Earth satellite, $k_{\oplus}$ = 0 074365, μ = 1(Earth radii)^3/(k^{-1} min)^2, P is in minutes, and a is in Earth equatorial radii (see Chap 2 of Ref 11, Sec 1 5 of Ref. 3, or p 493 of Ref. 8 for a discussion of this specialized system of units)

The remaining elements can now be obtained quite readily, provided the secular bulge perturbations which accumulate during the observation interval are not significant This is indeed the case for a lunar satellite

The determination of the preliminary orbit of a Martian satellite becomes considerably more complicated because the secular bulge perturbations are large For example, the line of nodes of a Martian satellite oıbit inclined 45° to the Martian equator and having a semimajor axis of 5300 km will rotate approximately 90° in 30 days This effect, which to the author's knowledge has not yet been included in preliminary orbit determination theories for aıtificial planetary satellites, has motivated the preparation of this paper

If the radial velocity curve deviates but little from a sine wave, an orbit of low eccentricity is indicated, and a circular preliminary orbit can be determined This orbit subsequently would be differentially corrected to introduce the eccentricity and the argument of perifocus A distorted radial velocity curve indicates an eccentric orbit, requiring an elliptical determination The circular procedure is considerably simpler than the elliptical form as the latter involves measurements of amplitudes and areas on the radial velocity curve Based on preliminary orbit determination experience for Earth satellites,[14] the determination of a circular preliminary orbit, followed by differential correction, probably could be used for orbits with eccentricities up to about 0 2

A numerical example was carried out to verify the theory for determining a preliminary circular orbit for a lunar satellite from range-rate observations It revealed that the observations used in the orbit determination must be selected with considerable care

II Preliminary Considerations

Basic Equation of the Problem

The fundamental equation of the problem is derived as follows The vector relationship between the positions of the observer, satellite, and the satellite's dynamical center can be written as (see Fig 1)

$$\boldsymbol{\rho} = \mathbf{r} + \mathbf{R} \tag{3}$$

where $\boldsymbol{\rho}$ is the position of the object with respect to the observer, $\mathbf{r}$ is the position of the satellite with respect to its dynamical center, and $\mathbf{R}$ is the position of the satellite's dynamical centeı with respect to the observer From Eq (3),

$$\boldsymbol{\rho} \cdot \boldsymbol{\rho} = \rho^2 = \mathbf{r} \cdot \mathbf{r} + 2\mathbf{r} \cdot \mathbf{R} + \mathbf{R} \cdot \mathbf{R} \tag{4}$$

Differentiation of Eq (4) gives

$$\rho\dot{\rho} = \mathbf{r} \cdot \dot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \mathbf{R} + \mathbf{r} \cdot \dot{\mathbf{R}} + \mathbf{R} \cdot \dot{\mathbf{R}} \tag{5}$$

Equation (5) represents the basic equation of the problem

In Eq (5), an approximation to the range ρ is the lunar or planetary distance from the observer $|\mathbf{R}|$, the radial velocity of the satellite with respect to the observer $\dot{\rho}$ is observed, and the position $\mathbf{R}$ and velocity $\dot{\mathbf{R}}$ of the dynamical center with respect to the observer can be computed from known data for any observation time The unknowns in Eq (5) are the satellite position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ with respect to the dynamical center These quantities are different for each observation time Sections III and IV will develop means of transforming these variables into functions of the two-body constants that define the orbit at some epoch Thus modified, a set of Eqs (5), one for each observation will be solved for the elements of the preliminary orbit

Position and Velocity of the Dynamical Center

For a lunar satellite (see Fig 1), $\mathbf{R}$ and $\dot{\mathbf{R}}$ aıe given by

$$\mathbf{R} = -\mathbf{r}_T + \mathbf{r}_{☾} \tag{6}$$

$$\dot{\mathbf{R}} = -\dot{\mathbf{r}}_T + \dot{\mathbf{r}}_{☾} \tag{7}$$

where $\mathbf{r}_T$ and $\dot{\mathbf{r}}_T$ denote the position and velocity of the observer with respect to the center of the Earth, and $\mathbf{r}_☾$ and $\dot{\mathbf{r}}_☾$ denote the geocentric position and velocity of the moon

For a planetary satellite, $\mathbf{R}$ and $\dot{\mathbf{R}}$ become (see Fig 1)

$$\mathbf{R} = -\mathbf{r}_T - \mathbf{r}_\oplus + \mathbf{r}_p \tag{8}$$

$$\dot{\mathbf{R}} = -\dot{\mathbf{r}}_T - \dot{\mathbf{r}}_\oplus + \dot{\mathbf{r}}_p \tag{9}$$

where $\mathbf{r}_\oplus$ and $\dot{\mathbf{r}}_\oplus$ are the heliocentric position and velocity of the earth, and $\mathbf{r}_p$ and $\dot{\mathbf{r}}_p$ are the heliocentric position and velocity of the planet $\mathbf{R}$ and $\dot{\mathbf{R}}$ can thus be computed for any observation time

Radial Velocity Curve

The effect of the observer's diurnal motion must be removed from the observed Doppler data before the observations are used to determine the period of the lunar or planetary satellite This is done by subtracting the term $\dot{\mathbf{r}}_T\ \mathbf{r}_{\oplus p}/r_{\oplus p}$ from the Doppler observations $\dot{\mathbf{r}}_T$ is the observer's geocentric velocity, and $\mathbf{r}_{\oplus p}$ is the geocentric position of the primary body The latter is obtained from data tabulated in the national ephemerides Actually, $\boldsymbol{\varrho}/\rho$ should be used instead of $\mathbf{r}_{\oplus p}/r_{\oplus p}$, but the former quantity is unknown at this stage of the problem

For a planetary satellite, the effect of the Earth's and planet's motion around the sun must also be removed from the observations To do this, the terms $\dot{\mathbf{r}}_\oplus\ \mathbf{r}_{\oplus p}/r_{\oplus p}$ and $\dot{\mathbf{r}}_p\ \mathbf{r}_{\oplus p}/r_{\oplus p}$, respectively, would be subtracted from the observations $\dot{\mathbf{r}}_\oplus$ and $\dot{\mathbf{r}}_p$ are the heliocentric velocities of the Earth and planet, and are available from data contained in the ephemerides For a lunar satellite, the quantity $\dot{\mathbf{r}}_☾\ \mathbf{r}_☾/r_☾$ is also removed from the observed data Again, $\mathbf{r}_☾/r_☾$ is being used to approximate $\boldsymbol{\varrho}/\rho$, since the latter is not known

Planetary Bulge Perturbations

The principle secular perturbations caused by the planetary bulge must be included in the determination of the preliminary planetary satellite orbit The secular variations in the longitude of the ascending node, the argument of perifocus, and the mean anomaly of a satellite orbit about an oblate planet, denoted by $\Omega^{\diagdown}$, $\omega^{\diagdown}$, and $M^{\diagdown}$, are, from general perturbations theory (Chap 19 of Ref 11, Ref 4, and Ref 15),

$$\Omega^{\diagdown} = -\tfrac{3}{2}J_2(a_p/p)^2 n \cos i \tag{10}$$

$$\omega^{\diagdown} = \tfrac{3}{4}J_2(a_p/p)^2 n(4 - 5\sin^2 i) \tag{11}$$

$$M^{\diagdown} = \tfrac{3}{2}J_2(a_p/p)^2 n(1 - e^2)^{1/2}(1 - \tfrac{3}{2}\sin^2 i) \tag{12}$$

where

$$\left.\begin{aligned}
J_2 &= \text{the coefficient of the second zonal harmonic in the planet's gravitational potential}\\
a_p &= \text{the equatorial radius of the planet}\\
p = a(1-e^2) &= \text{the semilatus rectum of the satellite orbit}\\
n = k\mu^{1/2}a^{-3/2} &= \text{the mean angular motion of the satellite}\\
i &= \text{the inclination of the satellite orbit plane to the planet's equator plane}\\
e &= \text{the eccentricity of the orbit}
\end{aligned}\right\} \tag{13}$$

For Martian satellite orbits, the constants required for Eqs (10–12) are[16] $(J_2)_♂ = 0\ 0020$, $k_♂ = 0\ 0621$, $a_♂ = 3415$ km, and $\mu = 1(\text{Mars' radii})^3/(k_♂^{-1}\ \text{min})^2$ Considering that the observations will extend over the order of perhaps several months, a numerical evaluation of Eqs (10–12) makes it apparent that the secular bulge induced variations must be included in the determination of the preliminary orbit of a Martian satellite For example, for $i = 45°$, $e = 0$, and $a = 5300$ km, Eq (10) shows that the secular change in Ω amounts to about 3 °1/day

The equations of motion for lunar satellites, including the perturbations due to the asphericity of the moon and the gravitational attractions of the sun and Earth, have been numerically integrated by Brumberg et al[5] They find the perturbative effects to be very small Schechter,[18] who applied general perturbations theory to determine the perturbative effects of the Earth and sun on a lunar satellite, also shows the effects are very small The perturbations on a lunar satellite can, therefore, be neglected in the preliminary orbit determination This greatly simplifies the determination of the orbit from Doppler data alone

III Circular First Approximation

An orbit of low eccentricity is indicated by a radial velocity curve, which closely resembles a sine wave A circular first approximation to the orbit is desirable in this case The circular first approximation is very simple in principle, and would be followed, of course, by a differential correction which would correct the circular orbit assumptions

For a circular orbit, the eccentricity $e = 0$ and the argument of perifocus ω is undefined Therefore, only four elements are required to define the orbit These will be taken to be embodied in the semimajor axis a, and the unit vectors $\mathbf{U}_0$ and $\mathbf{V}_0$ at some epoch t_0 $\mathbf{U}$ is the radial unit vector along $\mathbf{r}$ $\mathbf{V}$ also lies in the orbit plane, being perpendicular to $\mathbf{U}$ and pointed in the direction of motion (see Fig 2)

The components of $\mathbf{U}$ and $\mathbf{V}$, in terms of the true argument of latitude $u = v + \omega$, the longitude of the ascending node Ω, and the inclination i, are

$$\mathbf{U}\begin{cases} U_x = \cos u \cos\Omega - \sin u \sin\Omega \cos i\\ U_y = \cos u \sin\Omega + \sin u \cos\Omega \cos i\\ U = \sin u \sin i \end{cases} \tag{14}$$

$$\mathbf{V}\begin{cases} V_x = -\sin u \cos\Omega - \cos u \sin\Omega \cos i\\ V_y = -\sin u \sin\Omega + \cos u \cos\Omega \cos i\\ V_z = \cos u \sin i \end{cases} \tag{15}$$

Similar expressions can be written for $\mathbf{U}_0$ and $\mathbf{V}_0$ by replacing the two-body variable u with u_0 for two-body motion, and by also replacing the two-body constant Ω with Ω_0 when the secular bulge perturbations are being included

The time dependent position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ in the basic Eq (5) are now to be expressed in terms of the two-body constants a, $\mathbf{U}_0$, and $\mathbf{V}_0$, and, for a Martian satellite, their secular bulge induced variations It is assumed that the period P, and consequently the mean motion n, are known from the radial velocity curve With the mass of the primary body known, the semimajor axis a is also available from Eq (2)

Preliminary Circular Orbit Neglecting Perturbations

First then, from two-body considerations,

$$\mathbf{r} = a\mathbf{U} = a[\mathbf{U}_0 \cos n(t - t_0) + \mathbf{V}_0 \sin n(t - t_0)] \tag{16}$$

$$\dot{\mathbf{r}} = |\dot{\mathbf{r}}|\mathbf{V} = -(\mu/a)^{1/2}[\mathbf{U}_0 \sin n(t - t_0) - \mathbf{V}_0 \cos n(t - t_0)] \tag{17}$$

where the substitution $n(t - t_0) = (v - v_0)$ has been made, and

$$|\dot{\mathbf{r}}| = (\mu/a)^{1/2} \tag{18}$$

is the tangential velocity of the satellite Also for a circular orbit,

$$\mathbf{r}\ \dot{\mathbf{r}} = 0 \tag{19}$$

Substitution of Eqs (16, 17, and 19) into Eq (5) gives

$$\rho\dot{\rho} - \mathbf{R}\ \dot{\mathbf{R}} = \mathbf{U}_0[-(\mu/a)^{1/2}\mathbf{R} \sin n(t - t_0) + a\dot{\mathbf{R}} \cos n(t - t_0)] + \mathbf{V}_0[(\mu/a)^{1/2}\mathbf{R} \cos n(t - t_0) + a\dot{\mathbf{R}} \sin n(t - t_0)] \tag{20}$$

Equation (20) is rewritten in the form

$$A = U_{x0}C_x + U_{y0}C_y + U_0C + V_{x0}D_x + V_{y0}D_y + V_{z0}D \tag{21}$$

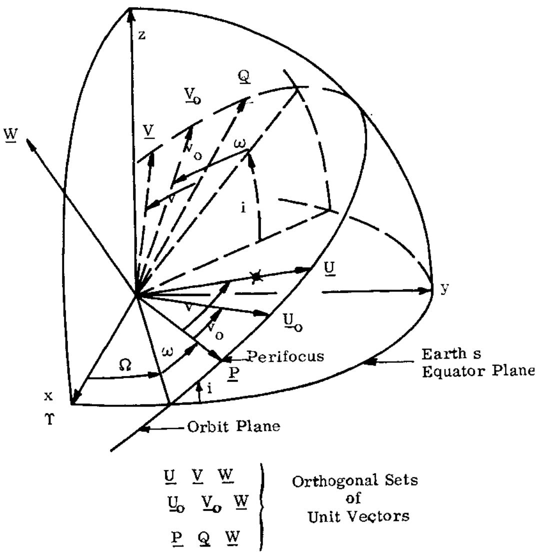

Fig 2 Orientation unit vectors and angles

where

$$A \triangleq \rho\dot{\rho} - \mathbf{R}\,\dot{\mathbf{R}} \tag{22}$$

$$\mathbf{C} \triangleq -(\mu/a)^{1/2}\mathbf{R}\sin n(t - t_0) + a\dot{\mathbf{R}}\cos n(t - t_0) \tag{23}$$

$$\mathbf{D} \triangleq (\mu/a)^{1/2}\mathbf{R}\cos n(t - t_0) + a\dot{\mathbf{R}}\sin n(t - t_0) \tag{24}$$

In Eq (21), the auxiliaries A, $\mathbf{C}$, and $\mathbf{D}$ represent quantities that are known for each observation time The components of $\mathbf{U}_0$ and $\mathbf{V}_0$ are the unknowns Numerical experimentation has shown that the solution to a set of Eqs (21) is sensitive to the data sampling pattern

The magnitude of $\mathbf{R}$ is used to approximate ρ, since $\mathbf{r}$ is not yet known Then a minimum of six range-rate observations will permit six Eqs (21) to be solved for the components of $\mathbf{U}_0$ and $\mathbf{V}_0$ which, along with a, constitute the elements of the lunar satellite circular orbit More than the minimum number of six observations will most likely be necessary to get a satisfactory solution, and the corresponding number of Eqs (21) would be solved by the method of least squares The observations of a lunar satellite will have to extend over several days to enable $\mathbf{R}$ and $\dot{\mathbf{R}}$ to change sufficiently so that the inversion of (21) will be possible

The conditions $\mathbf{U}_0\,\mathbf{U}_0 = \mathbf{V}_0\,\mathbf{V}_0 = 1$ and $\mathbf{U}_0\,\mathbf{V}_0 = 0$ could be used to reduce the unknowns and required observations to three However, the resulting nonlinear equations would be difficult to solve Also, there will be no lack of observations, since many more than the minimum number must be available before the range-rate curve can be established in the first place

Preliminary Circular Orbit Including Secular Bulge

The secular bulge perturbations accrued since the epoch time t_0 will now be incorporated into the formulation It is known from general perturbations theory that a and i for a satellite orbit have only periodic changes due to planetary flattening They will therefore be regarded as constants herein Also, the secular perturbation in $n(t - t_0)$ is $\dot{M}$, given by Eq (12) The secular perturbative variations in $\mathbf{U}_0$ and $\mathbf{V}_0$ are denoted by $\dot{\mathbf{U}}_0$ and $\dot{\mathbf{V}}_0$, respectively The satellite's position and velocity at time t are then

$$\mathbf{r} = a\{[\mathbf{U}_0 + (t - t_0)\dot{\mathbf{U}}_0]\cos[(n + \dot{M})(t - t_0)] + [\mathbf{V}_0 + (t - t_0)\dot{\mathbf{V}}_0]\sin[(n + \dot{M})(t - t_0)]\} \tag{25}$$

$$\dot{\mathbf{r}} = -(\mu/a)^{1/2}\{[\mathbf{U}_0 + (t - t_0)\dot{\mathbf{U}}_0]\sin[(n + \dot{M})(t - t_0)] - [\mathbf{V}_0 + (t - t_0)\dot{\mathbf{V}}_0]\cos[(n + \dot{M})(t - t_0)]\} \tag{26}$$

From Eqs (14) and (15), regarding i as constant, the components of $\dot{\mathbf{U}}_0$ and $\dot{\mathbf{V}}_0$ are

$$\dot{\mathbf{U}}_0\begin{cases}\dot{U}_{x0} = \dot{\Omega}(-U_{y0} - V_{x0}\cos i)\\ \dot{U}_{y0} = \dot{\Omega}(U_{x0} - V_{y0}\cos i)\\ \dot{U}_{z0} = \dot{\Omega}\,V_{z0}\cos i\end{cases} \tag{27}$$

$$\dot{\mathbf{V}}_0\begin{cases}\dot{V}_{x0} = \dot{\Omega}(-V_{y0} + U_{x0}\cos i)\\ \dot{V}_{y0} = \dot{\Omega}(V_{x0} + U_{y0}\cos i)\\ \dot{V}_{z0} = \dot{\Omega}\,U_{z0}\cos i\end{cases} \tag{28}$$

where the substitution

$$\dot{u}_0 = -\dot{\Omega}\cos i \tag{29}$$

has been made

Now, by substituting the foregoing Eqs (25–28, 22, and 19) into Eq (5), the basic equation to be solved for the components of $\mathbf{U}_0$ and $\mathbf{V}_0$ becomes

$$\begin{aligned}A = {} & U_{x0}[C_x + \dot{\Omega}(t - t_0)(C_y + D_x\cos i)] + \\ & U_{y0}[C_y + \dot{\Omega}(t - t_0)(-C_x + D_y\cos i)] + \\ & U_{z0}[C_z + \dot{\Omega}(t - t_0)D_z\cos i] + \\ & V_{x0}[D_x + \dot{\Omega}(t - t_0)(-C_x\cos i + D_y)] + \\ & V_{y0}[D_y + \dot{\Omega}(t - t_0)(-C_y\cos i - D_x)] + \\ & V_{z0}[D_z - \dot{\Omega}(t - t_0)C_z\cos i]\end{aligned} \tag{30}$$

where the auxiliary quantities $\mathbf{C}$ and $\mathbf{D}$ are now given by

$$\mathbf{C} \triangleq -(\mu/a)^{1/2}\mathbf{R}\sin[(n + \dot{M})(t - t_0)] + a\dot{\mathbf{R}}\cos[(n + \dot{M})(t - t_0)] \tag{31}$$

$$\mathbf{D} \triangleq (\mu/a)^{1/2}\mathbf{R}\cos[(n + \dot{M})(t - t_0)] + a\dot{\mathbf{R}}\sin[(n + \dot{M})(t - t_0)] \tag{32}$$

$\dot{\Omega}$ and $\dot{M}$ are given by Eqs (10) and (12), respectively The quantity A is defined by Eq (22), where ρ is approximated by $|\mathbf{R}|$ The components of $\mathbf{U}_0$, $\mathbf{V}_0$, $\mathbf{C}$, and $\mathbf{D}$ are referred to the planetocentric, for example the areocentric, coordinate system

The unknown inclination makes a set of Eqs (30) difficult to solve because of the manner in which i enters into $\mathbf{C}$ and $\mathbf{D}$ (through $\dot{M}$) By assuming a value for i, $\dot{\Omega}$ and the auxiliary quantities $\mathbf{C}$ and $\mathbf{D}$ can be computed Then six or more observed values for the range-rate will enable the corresponding number of Eqs (30) to be solved for the components of $\mathbf{U}_0$ and $\mathbf{V}_0$, which, along with a, embody the elements of the circular orbit A least-squares solution is again used with more than six observations For a Martian satellite, the observations may have to be spaced over a number of months before a set of Eqs (30) can be solved for the components of $\mathbf{U}_0$ and $\mathbf{V}_0$

Having solved for $\mathbf{U}_0$ and $\mathbf{V}_0$, the assumed value of i is compared with the value contained in the computed $\mathbf{U}_0$ and $\mathbf{V}_0$ The normal unit vector $\mathbf{W}$ (see Fig 2) is determined from

$$\mathbf{W} = \mathbf{U}_0 \times \mathbf{V}_0 \tag{33}$$

In particular,

$$W_z = U_{x0}V_{y0} - V_{x0}U_{y0} = \cos i \quad (0 \leq i < 180°) \tag{34}$$

The value of i as computed from Eq (34) is compared with the assumed value, or the value from a preceding approximation, and a new estimate is made The process is repeated until the assumed and computed values agree This then completes the circular first approximation

For an orbit of rather high inclination and, therefore, small secular bulge perturbations in Ω, the initial guess of i could possibly be obtained from two-body considerations according to Eqs (21, 33, and 34)

If desired, Ω with respect to the planet's equator plane and equinox can now be obtained from

$$\sin\Omega = W_x/\sin i \tag{35}$$

$$\cos\Omega = -W_y/\sin i \tag{36}$$

$$0° \leq \Omega < 360°$$

The time of the most recent crossing of the ascending node is

$$T\Omega = t_0 - (u_0/n) \tag{37}$$

where the true argument of latitude at the epoch is given by

$$\left.\begin{aligned} \sin u_0 &= U_0/\sin i \quad (38)\\ \cos u_0 &= V_0/\sin i \quad (39)\end{aligned}\right\} \quad 0° \leq u_0 < 360°$$

Note that there is no need for making measurements on the radial velocity curve (except for the determination of the period) in the approaches to the problem presented thus far The iteration involving the inclination enables the secular bulge perturbations to be included in a manageable form The importance of including the bulge perturbations is evident from Eq (30) where their buildup and influence on the solution for $\mathbf{U}_0$ and $\mathbf{V}_0$ after a long interval of time is apparent

The circular approximation could now be differentially corrected by any of a number of procedures currently being used for Earth satellites (e g , Sec 14L of Ref 11 and Ref 20) In these procedures, residuals in the range rate, that is, differences between the actual observations and those computed using the preliminary elements, are related to improvements to the preliminary elements by linear differential formulas

IV Elliptical First Approximation

An elliptical orbit will be indicated by a radial velocity curve which resembles a distorted sine wave There exist a number of possible approaches for determining the preliminary orbit in this case, all of which are more lengthy and involved than the circular preliminary orbit determination

A straightforward solution is afforded by applying portions of spectroscopic binary star theory to the problem This approach has the disadvantage, however, of requiring that certain measurements in addition to the determination of the period be made on the range-rate curve This may lead to difficulties when a low-altitude satellite is occulted by the primary body, since nearly half the range-rate curve may then be missing The curve will have to be approximated in this region, or the method of King[12] may possibly be used to construct the portion of the curve obscured by occultation Note that the maximum, minimum, and one of the zero relative radial velocity points will never be missing from the curve

Thus, one approach taken here will be to first find values for the eccentricity e and the mean anomaly M_0 at epoch t_0 from the range-rate curve by the method of Lehmann-Filhés, widely used for spectroscopic binary stars, or possibly by a harmonic analysis of the curve Even though the shape and amplitude of the curve will change with time, e and M_0, as determined from a curve based on observations extending over a limited number of revolutions, will not be affected Once the dimensional elements a, e, and M_0 have been found, the orientation elements i, Ω, and ω can quite readily be determined both with and without the inclusion of the secular bulge effects Deutsch[7] has developed a technique whereby three range-rate curves for a planetary satellite, several months apart, are used to determine i, Ω, and ω with respect to the plane of the ecliptic The bulge perturbations during this time are neglected, however

When the bulge perturbations are not considered, measurements of the range-rate curve need not necessarily be made Certain quantities can be assumed, and the solution can then be converged upon by iterative techniques

Spectroscopic Binary Star Summary

The observed radial velocity, as appropriately adjusted according to the discussion in Sec II, will be denoted by $\dot\rho^*$ From spectroscopic binary star theory (e g Chap XIV of Ref 19 and Ref 1), the radial velocity of a circumplanetary satellite can be expressed as

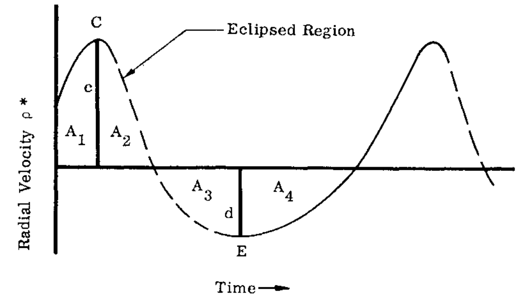

Fig 3 Typical radial velocity curve (e = 0 34)

$$\dot\rho^* = K\{e\cos\omega' + \cos[v + \omega']\} \tag{40}$$

where

$$K \triangleq (\mu/p)^{1/2}\sin i' \tag{41}$$

The orientation elements i' and ω' are referred to the plane of the sky and it is assumed that the orbit is always seen in the same direction Therefore Ω' remains undefined and the quadrant of i' is undetermined Equation (40) can be solved in a number of ways for e, M_0, ω', and $\sin i'$ (assuming the mass of the primary body is known; otherwise only the combination $a\sin i'$ can be determined)

According to the method of Lehmann-Filhés,

$$e\cos\omega' = (c - d)/(c + d) \tag{42}$$

$$e\sin\omega' = [2(cd)^{1/2}/(c + d)][(A_4 - A_2)/(A_4 + A_2)] \tag{43}$$

where c and d are amplitudes and A_4 and A_2 are areas on the range-rate curve A typical radial velocity curve illustrating these quantities is shown in Fig 3 The eccentricity is then

$$e = \{[e\cos\omega']^2 + [e\sin\omega']^2\}^{1/2} \tag{44}$$

An approximation to the time of perifocal passage T can also be found from the range-rate curve At perifocal passage, the true anomaly $v = 0$ Then, from Eq (40), the radial velocity at this point is

$$\dot\rho_T^* = K(1 + e)\cos\omega' \tag{45}$$

where e and $\cos\omega'$ are now known, and it can be shown that

$$K = \tfrac{1}{2}(c + d) \tag{46}$$

Two times on the $\dot\rho^*$ curve will correspond to the calculated $\dot\rho_T^*$ At point C (Fig 3), $\dot\rho^*$ is a maximum, and therefore $(v + \omega')$ must be 0 At point E, $\dot\rho^*$ is a minimum, and $(v + \omega')$ must be 180° Knowing ω', one can find v at points C and E, and the proper choice for the time of $\dot\rho_T^*$ can be made Knowing the time of perifocal passage, the mean anomaly at any epoch t_0 is

$$M_0 = n(t_0 - T) \tag{47}$$

Another approach to solving Eq (40) involves performing a harmonic analysis on the radial velocity curve rather than measuring areas Then σ_j and β_j in the Fourier representation of the curve,

$$\dot\rho^* = \Sigma\sigma_j\sin(jnt + \beta_j) \tag{48}$$

would be known By using Bessel functions to express the true anomaly v in terms of the mean anomaly M, Eq (40) becomes

$$\dot\rho^* = (2/e)K_1\Sigma J_j(je)\cos jM - 2K_2\Sigma J_j'(je)\sin jM \tag{49}$$

where the definitions

$$K_1 \triangleq K(1 - e^2)\cos\omega' \qquad K_2 \triangleq K(1 - e^2)^{1/2}\sin\omega' \tag{50}$$

have been made Plummer (pp 121, 122 of Ref 17) reduces Eqs (48) and (49) to

$$\frac{\sigma_2}{\sigma_1} + \left(\frac{e^3}{24} + \frac{e^5}{96}\right)\cos(4\beta_1 - 2\beta_2) = e - \frac{e^3}{4} \tag{51}$$

$$\frac{\sigma_2}{\sigma_1} \lambda \csc(4\beta_1 - 2\beta_2) = \frac{e^3}{24} + \frac{e^5}{96} \tag{52}$$

inclusive of e^5 terms The eccentricity is found from Eq (51) by iteration, and then Eq (52) gives λ, which is defined by

$$\lambda \triangleq nT + \beta_2 - \beta_1 \tag{53}$$

Equation (53) gives T and Eq (47) again gives M_0

Preliminary Elliptical Orbit Neglecting Perturbations

If e and M_0 have been found from measurements of amplitudes and areas on the range-rate curve, or from a harmonic analysis of the range-rate curve, the following procedure can be used to determine the orientation of the satellite orbit

The position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ are expressed in terms of components in the orbit plane Thus

$$\mathbf{r} = x_\omega \mathbf{P} + y_\omega \mathbf{Q} \tag{54}$$

$$\dot{\mathbf{r}} = \dot{x}_\omega \mathbf{P} + \dot{y}_\omega \mathbf{Q} \tag{55}$$

where the unit vectors $\mathbf{P}$ and $\mathbf{Q}$ are directed to perifocus (along x_ω axis) and 90° ahead of perifocus (along y_ω axis) (see Fig 2) The position and velocity components in the orbit plane are

$$x_\omega = a(\cos E - e) \tag{56}$$

$$y_\omega = a(1 - e^2)^{1/2} \sin E \tag{57}$$

$$\dot{x}_\omega = -[(\mu a)^{1/2}/r] \sin E \tag{58}$$

$$\dot{y}_\omega = [(\mu p)^{1/2}/r] \cos E \tag{59}$$

$$r = a(1 - e \cos E) \tag{60}$$

where E denotes the eccentric anomaly The components of $\mathbf{P}$ and $\mathbf{Q}$ in terms of the orientation angles i, Ω, and ω are

$$\mathbf{P}\begin{cases} P_x = \cos\omega \cos\Omega - \sin\omega \sin\Omega \cos i \\ P_y = \cos\omega \sin\Omega + \sin\omega \cos\Omega \cos i \\ P_z = \sin\omega \sin i \end{cases} \tag{61}$$

$$\mathbf{Q}\begin{cases} Q_x = -\sin\omega \cos\Omega - \cos\omega \sin\Omega \cos i \\ Q_y = -\sin\omega \sin\Omega + \cos\omega \cos\Omega \cos i \\ Q_z = \cos\omega \sin i \end{cases} \tag{62}$$

Substitution of Eqs (54) and (55) into the basic Eq (5) gives

$$\rho\dot{\rho} - \mathbf{R}\ \dot{\mathbf{R}} = r\dot{r} + \mathbf{P}\ \mathbf{S} + \mathbf{Q}\ \mathbf{T} \tag{63}$$

where the auxiliary quantities $\mathbf{S}$ and $\mathbf{T}$ are defined as

$$\mathbf{S} \triangleq \dot{x}_\omega \mathbf{R} + x_\omega \dot{\mathbf{R}} \tag{64}$$

$$\mathbf{T} \triangleq \dot{y}_\omega \mathbf{R} + y_\omega \dot{\mathbf{R}} \tag{65}$$

A set of Eqs (63) is then to be solved for the components of $\mathbf{P}$ and $\mathbf{Q}$ as follows

Given n, e, and M_0, the mean anomaly M and eccentric anomaly E can be calculated for each observation time from

$$M = M_0 + n(t - t_0) \tag{66}$$

$$M = E - e \sin E \tag{67}$$

Also

$$r\dot{r} = (\mu a)^{1/2} e \sin E \tag{68}$$

The auxiliary quantities $\mathbf{S}$ and $\mathbf{T}$ can now be computed for each observation time, and ρ is again approximated by $|\mathbf{R}|$

A set of Eqs (63) can thus be solved for the components of $\mathbf{P}$ and $\mathbf{Q}$ by using six or more range-rate observations More than the minimum six observations, along with a least-squares reduction, will probably be required to get a satisfactory solution The orientation angles can then be calculated from $\mathbf{P}$ and $\mathbf{Q}$

If e and M_0 are not found from measurements on the range-rate curve, it appears advantageous to express the position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ in terms of the position $\mathbf{r}_0$ and velocity $\dot{\mathbf{r}}_0$ at the epoch t_0 by means of

$$\mathbf{r} = f\mathbf{r}_0 + g\dot{\mathbf{r}}_0 \tag{69}$$

$$\dot{\mathbf{r}} = \dot{f}\mathbf{r}_0 + \dot{g}\dot{\mathbf{r}}_0 \tag{70}$$

where the classical f and g coefficients are (Chap 6 of Ref 10 and Ref 11)

$$f = 1 - (a/r_0)[1 - \cos(E - E_0)] \tag{71}$$

$$g = (a/\mu^{1/2})[(M - M_0) - (E - E_0) + \sin(E - E_0)] \tag{72}$$

$$\dot{f} = -[(\mu a)^{1/2}/rr_0] \sin(E - E_0) \tag{73}$$

$$\dot{g} = 1 - (a/r)[1 - \cos(E - E_0)] \tag{74}$$

and where

$$r_0 = a(1 - e \cos E_0) \tag{75}$$

$$(M - M_0) = n(t - t_0) = (E - E_0) - e \cos E_0 \sin(E - E_0) + e \sin E_0 [1 - \cos(E - E_0)] \tag{76}$$

$$r = a[1 - e \cos E_0 \cos(E - E_0) + e \sin E_0 \sin(E - E_0)] \tag{77}$$

Substitution of Eqs (69) and (70) into the basic Eq (5) gives

$$\rho\dot{\rho} - \mathbf{R}\ \dot{\mathbf{R}} = r\dot{r} + \mathbf{r}_0\ \mathbf{A} + \dot{\mathbf{r}}_0\ \mathbf{B} \tag{78}$$

where $\mathbf{A}$ and $\mathbf{B}$ are defined by

$$\mathbf{A} \triangleq \dot{f}\mathbf{R} + f\dot{\mathbf{R}} \tag{79}$$

$$\mathbf{B} \triangleq \dot{g}\mathbf{R} + g\dot{\mathbf{R}} \tag{80}$$

and where

$$r\dot{r} = \mathbf{r}\ \dot{\mathbf{r}} = (\mu a)^{1/2}[e \cos E_0 \sin(E - E_0) + e \sin E_0 \cos(E - E_0)] \tag{81}$$

Values are now assumed for the radial distance and radial velocity at the epoch, r_0 and $\dot{r}_0$, respectively Then the quantities $(e \cos E_0)$ and $(e \sin E_0)$ are computed from

$$e \cos E_0 = 1 - (r_0/a) \tag{82}$$

$$e \sin E_0 = r_0\dot{r}_0/(\mu a)^{1/2} \tag{83}$$

The quantity $(E - E_0)$ is found by iteration from Eq (76) for each observation time, enabling $\mathbf{A}$, $\mathbf{B}$, and $r\dot{r}$ to be computed A set of at least six Eqs (78) is solved for the components of $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ The iterative procedure would continue until

$$[x_0^2 + y_0^2 + z_0^2] - r_0^2 = 0 \tag{84}$$

$$(1/\mu)[\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2] - (2/r_0) + (1/a) = 0 \tag{85}$$

are satisfied The square brackets enclose the components of the calculated $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$, r_0 was approximated, and a is known This iterative procedure is similar to the approach suggested by Baker[2] for determining an Earth satellite preliminary orbit from only range observations

The preceding methods would be applicable to a lunar satellite since it does not undergo large perturbations during the observation interval For a Martian satellite, however, the secular bulge perturbations must be incorporated into the preliminary orbit determination scheme Accordingly, attention will now be directed to the inclusion of this effect

Preliminary Elliptical Orbit Including Secular Bulge Perturbations

The secular bulge perturbations are introduced into the elliptical first approximation through Eqs (54) and (55) At any time t

$$\mathbf{r} = [x_\omega + x_\omega^{\backprime}(t - t_0)][\mathbf{P} + \mathbf{P}^{\backprime}(t - t_0)] + [y_\omega + y_\omega^{\backprime}(t - t_0)][\mathbf{Q} + \mathbf{Q}^{\backprime}(t - t_0)] \quad (86)$$

$$\dot{\mathbf{r}} = [\dot{x}_\omega + \dot{x}_\omega^{\backprime}(t - t_0)][\mathbf{P} + \mathbf{P}^{\backprime}(t - t_0)] + [y_\omega + y_\omega^{\backprime}(t - t_0)][\mathbf{Q} + \mathbf{Q}^{\backprime}(t - t_0)] \quad (87)$$

where the grave symbol ($\backprime$) again denotes the secular perturbative variations

From the general perturbations theory, considering only secular variations, the following relationships are derived:

$$x_\omega^{\backprime} = 0 \quad (88)$$

$$\dot{x}_\omega^{\backprime} = \frac{3}{2}\mu J_2 \frac{a_p^2}{p^4}\left[\frac{1}{2e} + \frac{9}{4}e + \frac{11e^3}{16} + 0(e^4)\right]\left(1 - \frac{3}{2}\sin^2 i\right) \quad (89)$$

$$y_\omega^{\backprime} = -\frac{3}{2}(\mu p)^{1/2}\frac{J_2 a_p^2}{p^3}\left[\frac{1}{2e} + \frac{9}{8}e + \frac{11e^3}{32} + 0(e^4)\right] \times \left(1 - \frac{3}{2}\sin^2 i\right) \quad (90)$$

$$y_\omega^{\backprime} = 0 \quad (91)$$

Equations (61) and (62) give

$$\mathbf{P}^{\backprime}\begin{cases} P_x^{\backprime} = -\Omega^{\backprime} P_y + \omega^{\backprime} Q_x \\ P_y^{\backprime} = \Omega^{\backprime} P_x + \omega^{\backprime} Q_y \\ P_z^{\backprime} = \omega^{\backprime} Q \end{cases} \quad (92)$$

$$Q^{\backprime}\begin{cases} Q_x^{\backprime} = -\Omega^{\backprime} Q_y - \omega^{\backprime} P_x \\ Q_y^{\backprime} = \Omega^{\backprime} Q_x - \omega^{\backprime} P_y \\ Q^{\backprime} = -\omega^{\backprime} P \end{cases} \quad (93)$$

where $\Omega^{\backprime}$ and $\omega^{\backprime}$ are given by Eqs (10) and (11)

Substitution of Eqs (86) and (87) into the basic Fq (5), along with the auxiliary definitions

$$\mathbf{S} \triangleq x_\omega \dot{\mathbf{R}} + \dot{x}_\omega \mathbf{R} \quad (94)$$

$$T \triangleq y_\omega \dot{\mathbf{R}} + y_\omega \mathbf{R} \quad (95)$$

$$\psi \triangleq \frac{3}{2}\mu\frac{J_2 a_p^2}{p^3}\left(1 + \frac{3e^2}{2}\right)\left(1 - \frac{3}{2}\sin^2 i\right) = -r\dot{r}^{\backprime} \quad (96)$$

$$\gamma \triangleq \frac{3}{2}\mu\frac{J_2 a_p^2}{p^4}\left(\frac{1}{2e} + \frac{9}{4}e + \frac{11e^3}{16}\right)\left(1 - \frac{3}{2}\sin^2 i\right) \quad (97)$$

$$\nu \triangleq \frac{3}{2}(\mu p)^{1/2}\frac{J_2 a_p^2}{p^2}\left(\frac{1}{2e} + \frac{9}{8}e + \frac{11e^3}{32}\right)\left(1 - \frac{3}{2}\sin^2 i\right) \quad (98)$$

finally leads to

$$\begin{aligned} \rho\dot{\rho} - \mathbf{R}\,\dot{\mathbf{R}} = r\dot{r} - \psi(t - t_0) &+ \\ P_x[S_x + (S_y\Omega^{\backprime} - T_x\omega^{\backprime} + \gamma R_x)(t - t_0)] &+ \\ P_y[S_y + (-S_x\Omega^{\backprime} - T_y\omega^{\backprime} + \gamma R_y)(t - t_0)] &+ \\ P\,[S\; + (-T\,\omega^{\backprime} + \gamma R\,)(t - t_0)] &+ \\ Q_x[T_x + (T_y\Omega^{\backprime} + S_x\omega^{\backprime} - \nu\dot{R}_x)(t - t_0)] &+ \\ Q_y[T_y + (-T_x\Omega^{\backprime} + S_y\omega^{\backprime} - \nu\dot{R}_y)(t - t_0)] &+ \\ Q\,[T\; + (S\,\omega^{\backprime} - \nu\dot{R}\,)(t - t_0)]& \end{aligned} \quad (99)$$

The vector components are referred to the planetocentric, for example, the areocentric, coordinate system

Having found e and M_0 from measurements on the range-rate curve, the auxiliary quantities $\mathbf{S}$ and $\mathbf{T}$, along with $r\dot{r}$, can be computed for each observation time from Eqs (66–68 and 56–60)

As in the circular orbit case, an iteration on the inclination is suggested for solving a set of Eqs (99) With a minimum of six range-rate observations, or more than six and a least-squares reduction, and an assumed inclination, a set of Eqs (99) is solved for the components of $\mathbf{P}$ and $\mathbf{Q}$ at the epoch t_0 Then one obtains

$$\mathbf{W} = \mathbf{P} \times \mathbf{Q} \quad (100)$$

and in particular,

$$W_z = P_xQ_y - Q_xP_y = \cos i \; (0 \leq i < 180°) \quad (101)$$

The value of i from Eq (101) is compared with the assumed i, and the iteration is repeated until the two values agree

V Numerical Example

The final proof of any theory lies in how well it withstands an actual numerical application To this end, a circular lunar satellite orbit having $a = 1\ 08$ lunar radii was studied

The Doppler or range rate observations were first simulated for the fictitious lunar satellite Then the period was determined from the Doppler curve Finally, the remaining elements were recovered by means of the theory developed in Sec III A least-squares procedure was programmed to carry out the solution of a set of Eqs (21) for the components of $\mathbf{U}_0$ and $\mathbf{V}_0$

It was found that, for a lunar satellite, the solution to the set of Eqs (21) is very sensitive to the approximation $\rho \sim |\mathbf{R}|$ When no regard was given as to where along the orbit the observations were taken, the results were quite poor

By limiting the observations to the regions where $|\dot{\rho}|$ is maximum, that is, where the satellite is near the limb of the moon, the results were very good This is because the approximation $\rho \sim |\mathbf{R}|$ is best at these points The determination of the elements was also improved by extending the observation span over several days

Additional numerical experimentation with the theory would include an error analysis, to determine the effects of random and bias observation errors on the solution to a set of Eqs (21)

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Shadows Produced by Spin-Stabilized Communication Satellites

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A spin-stabilized satellite genei ally utilizes a so-called toroidal antenna that has a null cone surrounding the axis of symmetry (spin axis) The greatest gain over isotropic antennas occurs with the widest possible null cone However, wide null cones may cause shadows on the earth and prohibit communication between mutually visible ground stations and the satellite It is shown how the size and shape of the shadowed zones may be calculated for arbitrary orientations of the spin axis It is also shown how to find the narrowest possible antenna beam, which avoids any null-cone shadow on earth when the spin axis is maintained perpendicular to the ecliptic plane (desirable for thermal reasons) For near polar orbits, it is not possible to escape null-cone shadow and still realize any appreciable antenna gain if the spin axis is maintained exactly perpendicular to the ecliptic plane However, it is shown that a ieasonably small tilt of the spin axis will either reduce the shadow to an acceptable level or eliminate it entirely The critical tilt angles, for complete elimination of null-cone shadow, and the effect of nodal regression due to earth oblateness may be found from given curves

Nomenclature

A_p = projected area seen from satellite
A_{p0} = projected area of effective earth disk
A_p' = projected area of spherical cap (see Fig 5)
A = surface area of null-cone shadow
A_{0} = surface area of effective visible earth
A' = surface area of spherical cap (see Fig 5)
e_m = minimum elevation angle of antenna on earth
H = geocentric altitude of satellite
i = inclination of orbit referred to equator
i' = inclination or orbit referred to ecliptic
I_p = $A_p - A_p'$
I_s = $A - A_s'$
R = radius of earth
u = argument of latitude referred to equator
u' = argument of latitude referred to ecliptic
α_1 = angle subtended, at satellite, by effective earth limb
β = acute angle between spin axis and satellite earth line
β_1 = $\alpha_1 + \gamma$
β_m = minimum value of β anywhere in a given orbit
β_{0m} = value of β_m when spin axis is perpendicular to ecliptic
γ = semivertex angle of null cone
γ_0 = fixed null-cone angle
γ_{max} = $\beta_m - \alpha_1$
γ^* = $90 - \alpha_1$
ϵ = obliquity of ecliptic (taken as $23° 27'$)
θ = spherical coordinate (colatitude) of point where cone generator pierces earth surface
θ_1 = θ evaluated at σ_1
θ_2 = $90 - e_m - \alpha_1$
ν = $u - u'$
σ = angle between satellite earth line and generic cone generator
$\sigma_1\ \sigma_2$ = limit ing values of σ
τ = tilt angle of spin axis
τ = $\alpha_1 + \gamma_0 - \beta_{0m}$
Φ = spherical coordinate (longitude) of point where cone generator pierces earth surface
Ω = longitude of node measured along equator
Ω' = longitude of node measured along ecliptic

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1 Introduction

A NUMBER of papers[1–4] have been published on problems associated with earth coverage by communication satellites These have, for the most part, been concerned with the probability of mutual visibility between selected